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Use of Analytical Gradients to Calculate Optimal Gravity-Assist Trajectories

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A procedure for calculating the analytical derivatives required for an indirect optimization method of combined, long-duration finite burn and multiple gravity-assist trajectories is presented. The analytical derivatives are calculated using the state transition matrix associated with the complete set of the Euler–Lagrange equations of the optimal control problem on each trajectory segment as well as a state transition matrix that maps perturbations across any discontinuities in the state as a result of a zero-sphere-of-influence patched conic flyby or impulsive maneuver. As applications, the method is used to find Earth-to-Saturn trajectories that use combined low thrust and gravity assists and Earth-to-Saturn trajectories that use impulsive thrust and gravity assists. The state transition matrix derivatives are able to satisfy the trajectory constraints to orders of magnitude greater accuracy than the central difference derivatives. The state transition matrix method also requires less computational time to find an optimal trajectory.

Nomenclature

a	=	semimajor axis
C	=	constraint function
$\boldsymbol{g}$	=	gravitational acceleration acting on spacecraft
$\boldsymbol{h}$	=	orbital angular momentum of spacecraft
J	=	cost function
m	=	spacecraft mass
$P_{\max}$	=	maximum power available for propulsion system
$\boldsymbol{R}_{\text{fb}}$	=	transformation matrix used to express the $\Delta\boldsymbol{v}$ from the flyby in an inertial frame
$\boldsymbol{r}$	=	spacecraft position
r_m	=	periapse on hyperbolic flyby trajectory
t	=	time
$\boldsymbol{v}$	=	spacecraft velocity
$\boldsymbol{v}_p$	=	velocity of flyby planet
v_∞	=	velocity of spacecraft relative to flyby planet
$\boldsymbol{X}$	=	augmented spacecraft state (position, velocity, cost, and costates)
$\boldsymbol{Y}$	=	true state
$\boldsymbol{Y}^*$	=	nominal state
$\boldsymbol{y}$	=	perturbation of true state from nominal state
$\boldsymbol{Z}$	=	spacecraft state in orbital elements
α	=	adjoint control parameter
α'	=	adjoint control parameter
β	=	flyby angle
$\boldsymbol{\Gamma}$	=	thrust acceleration vector
γ	=	adjoint control parameter
γ'	=	adjoint control parameter
$\Delta\boldsymbol{v}$	=	impulsive velocity change
Δv_{pow}	=	impulsive velocity change made by the engine at periapse of the hyperbolic flyby
δ	=	flyby turn angle
κ	=	parameter that affects spacecraft state
λ_J	=	spacecraft cost costate
$\boldsymbol{\lambda}_r$	=	spacecraft position costate
$\boldsymbol{\lambda}_v$	=	spacecraft velocity costate
$\lambda^{\perp}_{v_{\text{fb}-}}$	=	component of velocity costate at $t_{\text{fb}-}$ that is perpendicular to $v_{\infty i}$
$\lambda^{\perp}_{v_{\text{fb}+}}$	=	component of velocity costate at $t_{\text{fb}+}$ that is perpendicular to $v_{\infty o}$
$\lambda^{\parallel}_{v_{\text{fb}-}}$	=	component of velocity costate at $t_{\text{fb}-}$ that is parallel to $v_{\infty i}$
$\lambda^{\parallel}_{v_{\text{fb}+}}$	=	component of velocity costate at $t_{\text{fb}+}$ that is parallel to $v_{\infty o}$
μ_{fb}	=	gravitational parameter of gravity assist planet
$\boldsymbol{\xi}$	=	Lagrange multiplier
$\boldsymbol{\Phi}$	=	state transition matrix for spacecraft state described by $\boldsymbol{X}$
$\boldsymbol{\Phi}_\alpha$	=	state transition matrix with orbital elements

Subscripts

$A0$	=	immediately after a flyby that occurs at t_{fb}
$B0$	=	at t_{fb} where no flyby occurs at t_{fb}
$B1$	=	immediately before a flyby that occurs at $t_{\text{fb}} + \Delta t_{\text{fb}}$
$C0$	=	state spacecraft must have at t_{fb} in order to pass through $\boldsymbol{X}_{C1}$
$C1$	=	immediately after a flyby that occurs at $t_{\text{fb}} + \Delta t_{\text{fb}}$
f	=	final
fb−	=	immediately before the flyby
fb+	=	immediately after the flyby
i	=	inbound
o	=	outbound
pfb	=	postflyby
0	=	initial
&	=	denotes a partial derivative mapped to a common time

Introduction

THE problem of determining optimal trajectories with multiple gravity assists has been addressed by numerous authors. D'Amario et al.[1] and Sauer[2] use the state transition matrix to calculate partial derivatives that are used to optimize an impulsive $\Delta\boldsymbol{v}$ multiple flyby trajectory. More recently Byrnes and Bright[3] and others have sought to remove the approximations caused by the patched conic technique used by previous researchers. Recent studies have sought optimal gravity-assist trajectories using finite-duration thrust arcs.[4,5] This paper presents a hybrid method using a continuous

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control to solve for optimal trajectories with a finite-thrust engine. The kinematic boundary conditions are satisfied through the use of equality constraints, and the transversality conditions are satisfied by minimizing the cost function using a sequential-quadratic-programming code. This method requires one to determine the derivatives of the cost and constraint functions with respect to all of the free parameters. The accuracy with which the optimal trajectory can be found depends on the accuracy of these derivatives. Most previous research has used finite difference methods to estimate the value of these derivatives because of the ease of calculating them. In this paper a method using the state transition matrix to calculate analytic derivatives for multiple gravity-assist trajectories will be demonstrated. These derivatives will be shown to be exact to the accuracy of the integrator used to propagate the equations of motion and the state transition matrix.

Impulsive $\Delta \boldsymbol{v}$ Model

For the impulsive $\Delta \boldsymbol{v}$ model, the trajectory begins with the spacecraft on an Earth escape trajectory with a given hyperbolic excess velocity vector. The magnitude and direction of this hyperbolic excess velocity are free parameters that are optimized. The spacecraft then coasts until the time of the first flyby. An impulsive Δv can be made parallel to the velocity of the spacecraft with respect to the flyby planet at periapse of the flyby. After the first flyby the spacecraft coasts until it makes an impulsive postflyby $\Delta \boldsymbol{v}$ maneuver and coasts until it makes its next flyby. The spacecraft can repeat the preceding step for any number of gravity assists. Following its final flyby, the spacecraft coasts until it makes a final impulsive $\Delta \boldsymbol{v}$ maneuver after which it coasts until the final time where it must intercept the target. The number of gravity assists and the planet providing each gravity assist must be determined a priori. This method determines only a locally optimal trajectory for a given sequence of gravity assists. The free parameters for this case are the initial time, final time, flyby times, postflyby maneuver times, flyby radii, flyby angles, magnitude and direction of the initial hyperbolic excess velocity, magnitude and direction of each impulsive $\Delta \boldsymbol{v}$, and magnitude of the impulsive $\Delta \boldsymbol{v}$ provided by the engine at periapse of each flyby. The cost is the sum of the impulsive Δv plus the initial hyperbolic excess velocity. The constraints require the spacecraft to have the same position as the planet providing each gravity assist at the time of the gravity assist. The spacecraft must also intercept the target planet at the final time.

The spacecraft state is numerically integrated using DLSODE* on each trajectory segment. DLSODE uses predictor–corrector methods for nonstiff ordinary differential equations and backward-differentiation formula methods for stiff differential equations. The Jet Propulsion Laboratory (JPL) DE405† ephemerides are used to determine the position of the sun and the planets. The gravity model used to propagate the equations of motion treats the sun as the center of an inertial frame and the only gravitational body that affects the motion of the spacecraft except for the zero-sphere-of-influence patched conic $\Delta \boldsymbol{v}$, which are obtained from the gravity assists.

Finite-Thrust Engines

The finite-thrust trajectory consists of a number of segments equal to one plus the number of gravity assists. The first segment begins after the spacecraft has escaped the initial planet on a parabolic escape trajectory. On the first segment the spacecraft travels from the initial planet to the first flyby planet. On each subsequent segment the spacecraft travels from one flyby planet to the next except for the last segment when the spacecraft travels from the final gravity-assist planet to the target. Figure 1 shows a single gravity-assist trajectory for the finite-thrust engine. The number of gravity assists and the planet providing each gravity assist must be determined a priori. This method only determines a locally optimal trajectory for a given sequence of gravity assists.

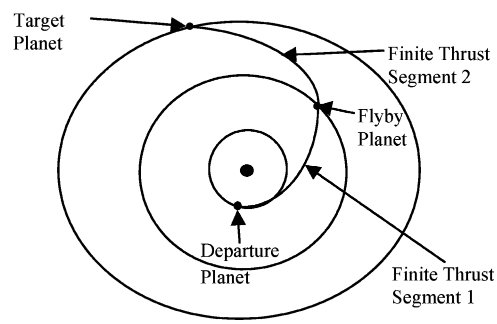

Fig. 1 Finite-thrust single flyby trajectory.

The optimal control problem is solved as a parameter optimization problem by treating the costate Lagrange multiplier vector as part of the parameter vector. The Euler–Lagrange equations are numerically integrated, and the control satisfies the Pontryagin maximum principle. The free parameters for this case are the initial time, final time, flyby time(s), flyby radii, flyby angles, and the initial Lagrange multipliers for each segment. The spacecraft is constrained to intercept each of the flyby planets at the time of each flyby and the target planet at the final time. The cost function is the total mass consumed.

To minimize the propellant consumed for a mission scenario using a power-limited-propulsion (PLP) engine, one can minimize the integral of the thrust acceleration squared, which is defined by Eq. (1). The thrust acceleration vector must be the velocity costate vector in order to satisfy Pontryagin's maximum principle. Minimizing J instead of minimizing the propellant consumed allows one to determine the optimal trajectory without knowing the initial mass of the spacecraft or the power available to the propulsion system. This model assumes that the power is constant throughout the mission. The mass of the spacecraft at any time can be determined using Eq. (3) (Ref. 6).

$$J \equiv \int_{t_0}^{t_f} \frac{\Gamma^2}{2}\,\mathrm{d}t \tag{1}$$

$$\boldsymbol{\Gamma} = \boldsymbol{\lambda}_v \tag{2}$$

$$m(t) = \frac{m(t_0)P_{\max}}{P_{\max} + m(t_0)J(t)} \tag{3}$$

The spacecraft state and its derivative with respect to time are defined to be

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ J \\ \boldsymbol{\lambda}_r \\ \boldsymbol{\lambda}_v \\ \lambda_J \end{pmatrix} \tag{4}$$

$$\boldsymbol{X}' = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{g}(\boldsymbol{r}) + \boldsymbol{\lambda}_v \\ \dfrac{\lambda_v^2}{2} \\ -\dfrac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}}\boldsymbol{\lambda}_v \\ -\boldsymbol{\lambda}_r \\ 0 \end{pmatrix} \tag{5}$$

The partial derivative of $\boldsymbol{X}'$ with respect to $\boldsymbol{X}$ for the PLP engine is given by Eq. (6). For the impulsive $\Delta \boldsymbol{v}$ model the derivative is the

*Data available online at http://www.llnl.gov/CASC/odepack [cited 28 August 2003].

†Data available online at http://ssd.jpl.nasa.gov/iau-comm4/de405iom/de405iom.ps [cited 22 July 2003].

square matrix of dimension six in the upper-left portion of Eq. (6):

$$\frac{\partial \boldsymbol{X}'}{\partial \boldsymbol{X}} = \begin{pmatrix} \boldsymbol{0}_{3x3} & \boldsymbol{I}_{3x3} & \boldsymbol{0}_{3x1} & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x1} \\ \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}} & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x1} & \boldsymbol{0}_{3x3} & \boldsymbol{I}_{3x3} & \boldsymbol{0}_{3x1} \\ \boldsymbol{0}_{1x3} & \boldsymbol{0}_{1x3} & 0 & \boldsymbol{0}_{1x3} & \boldsymbol{\lambda}_v^T & 0 \\ -\frac{\partial}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}}\boldsymbol{\lambda}_v & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x1} & \boldsymbol{0}_{3x3} & -\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{r}} & \boldsymbol{0}_{3x1} \\ \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x1} & -\boldsymbol{I}_{3x3} & \boldsymbol{0}_{3x3} & \boldsymbol{0}_{3x1} \\ \boldsymbol{0}_{1x3} & \boldsymbol{0}_{1x3} & 0 & \boldsymbol{0}_{1x3} & \boldsymbol{0}_{1x3} & 0 \end{pmatrix} \tag{6}$$

Adjoint Control Transformation

The values of the position and velocity costates at the beginning of each trajectory segment are free parameters that must be optimized in order to determine the optimal trajectory. To obtain an initial estimate for these costates, an adjoint control transformation is used.[7,8] A frame is defined such that the first axis points along the spacecraft velocity, the third axis points along the orbital angular momentum vector of the spacecraft, and the second axis is defined so the system is right handed.

$$\hat{e}_1 = \boldsymbol{v}/v, \qquad \hat{e}_2 = (\boldsymbol{h} \times \boldsymbol{v})/|\boldsymbol{h} \times \boldsymbol{v}|, \qquad \hat{e}_3 = \boldsymbol{h}/h \tag{7}$$

The adjoint control transformation shown in Fig. 2 allows one to estimate the magnitude and direction of the costates in this vehicle-centered frame instead of requiring one to estimate the costates in the inertial Cartesian frame where the problem is solved. The initial costate can thus be written as in Eq. (8). A transformation matrix can then be defined to express the costates on the inertial x, y, z bases as in Eqs. (10) and (11). Instead of estimating the initial value of the costates, the value of the adjoint control related variables α, γ, α', γ', $\lambda_v(t_0)$, and $\lambda'_v(t_0)$ can be estimated. It is easier to determine an initial estimate of these parameters than it is to estimate the initial costates.[7,8]

$$\boldsymbol{\lambda}_v(t_0) = \lambda_v(t_0) \begin{bmatrix} \cos(\alpha)\cos(\gamma) \\ \sin(\alpha)\cos(\gamma) \\ \sin(\gamma) \end{bmatrix} \begin{matrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{matrix} \tag{8}$$

$$\boldsymbol{R} = \begin{bmatrix} \frac{\boldsymbol{v}\cdot\hat{x}}{v} & \frac{(\boldsymbol{h}\times\boldsymbol{v})\cdot\hat{x}}{|\boldsymbol{h}\times\boldsymbol{v}|} & \frac{\boldsymbol{h}\cdot\hat{x}}{h} \\ \frac{\boldsymbol{v}\cdot\hat{y}}{v} & \frac{(\boldsymbol{h}\times\boldsymbol{v})\cdot\hat{y}}{|\boldsymbol{h}\times\boldsymbol{v}|} & \frac{\boldsymbol{h}\cdot\hat{y}}{h} \\ \frac{\boldsymbol{v}\cdot\hat{z}}{v} & \frac{(\boldsymbol{h}\times\boldsymbol{v})\cdot\hat{z}}{|\boldsymbol{h}\times\boldsymbol{v}|} & \frac{\boldsymbol{h}\cdot\hat{z}}{h} \end{bmatrix} \tag{9}$$

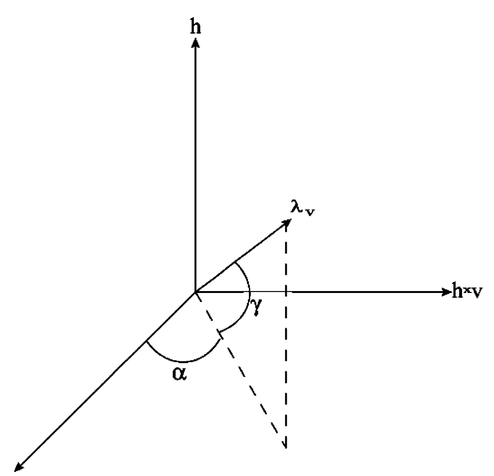

Fig. 2 Adjoint control transformation.

$$\boldsymbol{\lambda}_v(t_0) = \lambda_v(t_0)\boldsymbol{R} \begin{bmatrix} \cos(\alpha)\cos(\gamma) \\ \sin(\alpha)\cos(\gamma) \\ \sin(\gamma) \end{bmatrix} \begin{matrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{matrix} \tag{10}$$

$$\boldsymbol{\lambda}_r(t_0) = \left\{ \lambda_v(t_0)\boldsymbol{R} \begin{bmatrix} \alpha'\sin(\alpha)\cos(\gamma) + \gamma'\cos(\alpha)\sin(\gamma) \\ -\alpha'\cos(\alpha)\cos(\gamma) + \gamma'\sin(\alpha)\sin(\gamma) \\ -\gamma'\cos(\gamma) \end{bmatrix} - \lambda'_v(t_0)\boldsymbol{R} \begin{bmatrix} \cos(\alpha)\cos(\gamma) \\ \sin(\alpha)\cos(\gamma) \\ \sin(\gamma) \end{bmatrix} - \lambda_v(t_0)\boldsymbol{R}' \begin{bmatrix} \cos(\alpha)\cos(\gamma) \\ \sin(\alpha)\cos(\gamma) \\ \sin(\gamma) \end{bmatrix} \right\} \begin{matrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{matrix} \tag{11}$$

Gravity-Assist Model

The flyby is modeled as a zero-sphere-of-influence patched conic, where the velocity of the spacecraft changes instantaneously and the position of the spacecraft remains constant. The spacecraft is constrained to intercept the center of the flyby planet at the time of the gravity assist even though the flyby radius is nonzero. The cost difference of targeting the flyby radius instead of the center of the planet is minimal. The impulsive thrust engine is able to create a Δv parallel to v_∞ at the location that would be periapse of an unpowered flyby. Because the finite-thrust engines are unable to provide an impulsive velocity change, Δv_{pow} is equal to zero for the finite-thrust engine model. As a result, J is constant across the flyby. The flyby angle, β shown in Fig. 3, is the angle from the $(v_{\infty i} \times v_p) \times v_{\infty i}$ axis to the projection of $v_{\infty o}$ in the plane defined by the $(v_{\infty i} \times v_p) \times v_{\infty i}$ and $v_{\infty i} \times v_p$ axes. This angle determines which side of the planet the spacecraft flies past during the gravity assist. The flyby radius can be constrained to be greater than or equal to a minimum value. The velocity change during the flyby can be determined from Eqs. (12–17).

$$\sin\left(\frac{\delta_1}{2}\right) = \frac{\mu_{\text{fb}}}{\mu_{\text{fb}} + r_m v_{\infty_i}^2} \tag{12}$$

$$\sin\left(\frac{\delta_2}{2}\right) = \frac{\mu_{\text{fb}}}{\mu_{\text{fb}} + r_m v_{\infty_o}^2} \tag{13}$$

$$v_{\infty_o} = v_{\infty_i} + \Delta v_{\text{pow}} \tag{14}$$

$$\delta = \frac{\delta_1 + \delta_2}{2} \tag{15}$$

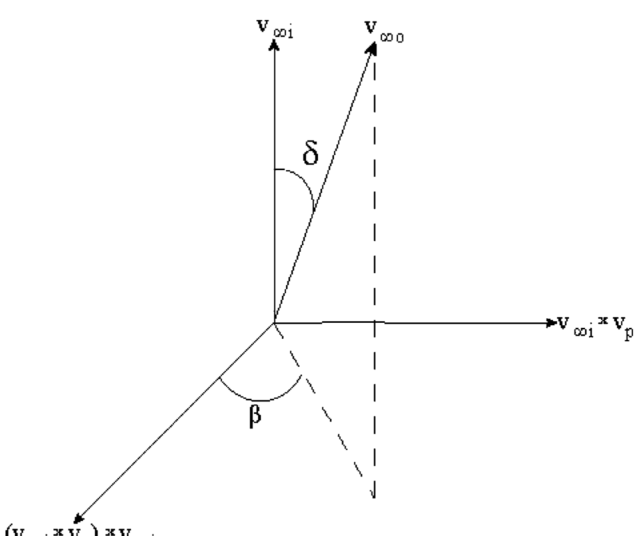

Fig. 3 Flyby angle and flyby turn angle.

$$
\boldsymbol{R}_{\mathrm{fb}} = \begin{bmatrix} \dfrac{\boldsymbol{v}_{\infty_i} \cdot \hat{x}}{v_{\infty_i}} & \dfrac{\left[\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right] \cdot \hat{x}}{\left|\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right|} & \dfrac{\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \cdot \hat{x}}{\left|\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right|} \\ \dfrac{\boldsymbol{v}_{\infty_i} \cdot \hat{y}}{v_{\infty_i}} & \dfrac{\left[\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right] \cdot \hat{y}}{\left|\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right|} & \dfrac{\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \cdot \hat{y}}{\left|\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right|} \\ \dfrac{\boldsymbol{v}_{\infty_i} \cdot \hat{z}}{v_{\infty_i}} & \dfrac{\left[\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right] \cdot \hat{z}}{\left|\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \times \boldsymbol{v}_{\infty_i}\right|} & \dfrac{\left(\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right) \cdot \hat{z}}{\left|\boldsymbol{v}_{\infty_i} \times \boldsymbol{v}_p\right|} \end{bmatrix} \tag{16}
$$

$$
\boldsymbol{v}(t_{\mathrm{fb}+}) = \boldsymbol{v}_p(t_{\mathrm{fb}}) + v_{\infty_o}\boldsymbol{R}_{\mathrm{fb}} \begin{bmatrix} \cos(\delta) \\ \sin(\delta)\cos(\beta) \\ \sin(\delta)\sin(\beta) \end{bmatrix} \tag{17}
$$

The costates are not continuous across the instantaneous gravity assist. The optimal value for the costates after an unpowered flyby can be determined from Eqs. (18–21) and the fact that the velocity costate vector immediately before and immediately after the flyby is in the plane defined by the incoming and outgoing hyperbolic excess velocity vectors.[9] The Lagrange multiplier ξ's value is determined by solving the optimal control problem. In Eq. (21), $\boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]$ is defined to be the gravitational acceleration of the spacecraft at the time of the flyby. Because this is a zero-sphere-of-influence patched conic approximation, the gravitational force caused by the flyby planet is not included in $\boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]$. Also note that $\{[\boldsymbol{v}'_p(t_{\mathrm{fb}})] - \boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]\}$ would equal zero if the flyby planet had its equations of motion determined by integrating through the same gravity field that the spacecraft is traversing. If, however, the planet being flown by has its position and velocity determined by an ephemeris file that accounts for perturbations not included in the gravity field being used to determine the equations of motion of the spacecraft, this term might not equal zero.

$$
\boldsymbol{\lambda}_{r_{\mathrm{fb}-}} = \boldsymbol{\lambda}_{r_{\mathrm{fb}+}} + \boldsymbol{\xi} \tag{18}
$$

$$
\lambda^{\perp}_{v_{\mathrm{fb}-}} = \left|\boldsymbol{\lambda}_{v_{\mathrm{fb}-}} \times \boldsymbol{v}_{\infty_i}\right| = \left|\boldsymbol{\lambda}_{v_{\mathrm{fb}+}} \times \boldsymbol{v}_{\infty_o}\right| = \lambda^{\perp}_{v_{\mathrm{fb}+}} \tag{19}
$$

$$
\lambda^{\parallel}_{v_{\mathrm{fb}+}} - \lambda^{\parallel}_{v_{\mathrm{fb}-}} = \frac{4r_m(\lambda v^{\perp})[\sin(\delta/2)]^2}{\left(\mu\{1-[\sin(\delta/2)]^2\}^{\frac{1}{2}}\right)} \tag{20}
$$

$$
\boldsymbol{\lambda}^T_{r_{\mathrm{fb}+}}\boldsymbol{v}_{\infty_o} - \boldsymbol{\lambda}^T_{r_{\mathrm{fb}-}}\boldsymbol{v}_{\infty_i} = \left(\boldsymbol{\lambda}_{v_{\mathrm{fb}+}} - \boldsymbol{\lambda}_{v_{\mathrm{fb}-}}\right)^T\{[\boldsymbol{v}'_p(t_{\mathrm{fb}})] - \boldsymbol{g}[\boldsymbol{r}(t_{\mathrm{fb}})]\} \tag{21}
$$

Instead of using Eqs. (18–21), the values of the costatess immediately after the flyby can be added to the list of free parameters. The value of the costates on the minimum fuel transfer will satisfy these equations. Again an adjoint control transformation is used, and the optimal values of α, γ, α', γ', $\lambda_v(t_{\mathrm{fb}+})$, and $\lambda'_v(t_{\mathrm{fb}+})$ must be estimated. The costates immediately after the flyby are then defined according to Eqs. (10) and (11). As a result, these costates are a function of $\boldsymbol{r}(t_{\mathrm{fb}+})$ and $\boldsymbol{v}(t_{\mathrm{fb}+})$. Because $\boldsymbol{r}(t_{\mathrm{fb}+})$ and $\boldsymbol{v}(t_{\mathrm{fb}+})$ are a function of $\boldsymbol{r}(t_{\mathrm{fb}-})$ and $\boldsymbol{v}(t_{\mathrm{fb}-})$, the costates after the flyby are a function of $\boldsymbol{r}(t_{\mathrm{fb}-})$ and $\boldsymbol{v}(t_{\mathrm{fb}-})$; they are not however a function of $\boldsymbol{\lambda}_r(t_{\mathrm{fb}-})$ or $\boldsymbol{\lambda}_v(t_{\mathrm{fb}-})$.

Partial Derivatives with the State Transition Matrix

One way to calculate partial derivatives in a trajectory optimization algorithm is to use a finite difference method. Equation (22) provides a first- and second-order method to calculate the finite difference derivatives of the state at time t_1 with respect to a parameter κ, which affects the state at t_0. Figure 4 shows the definitions of $\boldsymbol{X}^+$ and $\boldsymbol{X}^-$. Error will be introduced into the derivative because, in general, $\boldsymbol{X}(t_1)$ will be corrupted by error associated with numerically integrating $\boldsymbol{X}$ from t_0 to t_1. In addition to this error source, the choice of $\Delta\kappa$ will introduce error as well. If $\Delta\kappa$ is too large, truncation error will be large; if $\Delta\kappa$ is too small, the derivatives will be corrupted by round-off error. If the proper perturbation size is used, a good estimate of the accuracy of a derivative of a function with respect to a parameter is one-half the accurate digits to which the function is known.[10] For this case it would be one-half of the accurate digits to which $\boldsymbol{X}(t_1)$ could be determined. Attempts to find the optimal perturbation size are complicated by the fact that the optimal value for the perturbation will be different not only for each derivative but for each derivative on each iteration. It is important to compute each derivative accurately because the convergence of any parameter optimization algorithm will be limited by the accuracy of the worst derivative.[11]

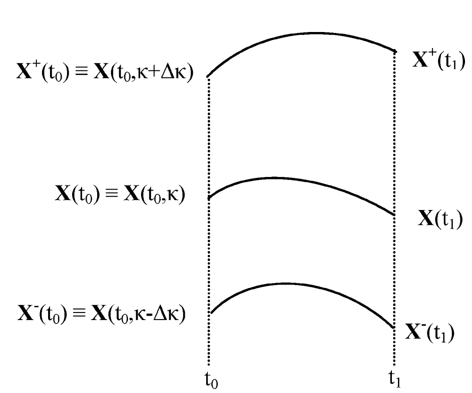

Fig. 4 Finite difference derivatives.

$$
\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} = \frac{\left\{\boldsymbol{X}^+(t_0, \kappa+\Delta\kappa) + \int_{t_0}^{t_1}\boldsymbol{X}'[\boldsymbol{X}^+(t), t]\,\mathrm{d}t\right\} - \left\{\boldsymbol{X}(t_0, \kappa) + \int_{t_0}^{t_1}\boldsymbol{X}'[\boldsymbol{X}(t), t]\,\mathrm{d}t\right\}}{\Delta\kappa}
$$

$$
\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} = \frac{\left\{\boldsymbol{X}^+(t_0, \kappa+\Delta\kappa) + \int_{t_0}^{t_1}\boldsymbol{X}'[\boldsymbol{X}^+(t), t]\,\mathrm{d}t\right\} - \left\{\boldsymbol{X}^-(t_0, \kappa-\Delta\kappa) + \int_{t_0}^{t_1}\boldsymbol{X}'[\boldsymbol{X}^-(t), t]\,\mathrm{d}t\right\}}{2\Delta\kappa} \tag{22}
$$

The state transition matrix provides an alternative method to calculate partial derivatives. It is a tool that maps the first-order perturbations in the complete spacecraft state (including costates) from one time to another without any error if the state transition matrix can be computed analytically. In general, the state transition matrix must be determined through numerical integration, which will introduce error into the mapping of these first-order perturbations. Assuming the analytical partial derivative of $\boldsymbol{X}(t_0)$ with respect to κ is known, one can use a Taylor series to calculate $X^+(t_0)$. The first-order part of the perturbation, $X^+(t_0) - X(t_0)$, can be mapped forward to the time t_1 using the state transition matrix. The partial derivative that is desired can then be calculated exactly if the analytic state transition matrix is known using the definition of a derivative shown in Eq. (23). In general these state transition matrix derivatives will not be exact because the state transition matrix will be numerically integrated, but these derivatives will be more accurate than the finite difference derivatives because truncation and round-off error will not be introduced into the derivatives using this method because the derivatives are taken in the limit as $\Delta\kappa$ goes to zero, and no difference is computed as in the finite difference case.

$$
\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} = \lim_{\Delta\kappa \to 0} \frac{\boldsymbol{X}^+(t_1) - \boldsymbol{X}(t_1)}{\Delta\kappa} \tag{23}
$$

To illustrate the calculation of a derivative using the state transition matrix, consider the following problem. The difference between

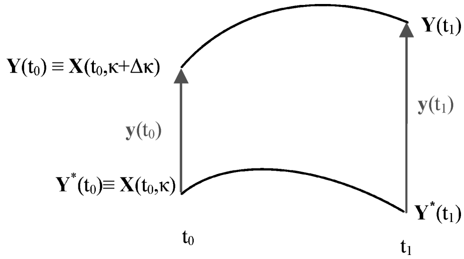

Fig. 5 State transition matrix derivative mappings.

the true state $\boldsymbol{Y}$ and the nominal state $\boldsymbol{Y}^*$ is defined to be $\boldsymbol{y}$ as shown in Eq. (24). From the definitions of $\boldsymbol{y}$, $\boldsymbol{Y}$, and $\boldsymbol{Y}^*$, Eq. (25) can be obtained using a Taylor series. The state transition matrix is defined so that Eqs. (26) and (27) are satisfied and its initial value is identity.[12] The state transition matrix for this problem will be 14×14 for the finite-thrust case because it includes the costates.[13]

$$\boldsymbol{y}(t) \equiv \boldsymbol{Y}(t) - \boldsymbol{Y}^*(t) \tag{24}$$

$$\boldsymbol{y}'(t) = \left(\frac{\partial \boldsymbol{Y}'}{\partial \boldsymbol{Y}}\right)_{Y = Y^*(t)} \boldsymbol{y}(t) + \mathcal{O}(\boldsymbol{y}^2) \tag{25}$$

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{y}(t_0) + \mathcal{O}\left[\boldsymbol{y}(t_0)\boldsymbol{y}^T(t_0)\right] \tag{26}$$

$$\boldsymbol{\Phi}'(t, t_0) = \left(\frac{\partial \boldsymbol{Y}'}{\partial \boldsymbol{Y}}\right)_{Y = Y^*(t)} \boldsymbol{\Phi}(t, t_0) \tag{27}$$

Equation (26) is valid as long as $\boldsymbol{Y}'[\boldsymbol{Y}(t), t]$ is a continuous function of $\boldsymbol{Y}$ and t. If this is not the case, additional terms must be added to these equations in order to determine the partial derivatives. To illustrate, consider a case where the value of the spacecraft state at time t_0 is a function of κ; κ is not a function of t_0; the partial derivative of $\boldsymbol{X}(t_0)$ with respect to κ is known; and $\boldsymbol{X}'[\boldsymbol{X}(t), t]$ is not an explicit function of κ. Define a perturbed trajectory to have initial condition $\boldsymbol{Y}(t_0) = \boldsymbol{X}(t_0, \kappa + \Delta\kappa)$, and define the nominal trajectory to have initial condition $\boldsymbol{Y}^*(t_0) = \boldsymbol{X}(t_0, \kappa)$ as seen in Fig. 5. Because the derivative of $\boldsymbol{X}(t_0)$ with respect to κ is known, one can write Eq. (28) using a Taylor series. Using Eqs. (26) and (28), $\boldsymbol{y}(t_1)$ can be determined as in Eq. (29). The partial derivative of $\boldsymbol{X}(t_1)$ with respect to κ is by definition given by Eq. (30), which leads to Eq. (31).

$$\boldsymbol{y}(t_0) \equiv \boldsymbol{Y}(t_0) - \boldsymbol{Y}^*(t_0) = \Delta\kappa \frac{\partial \boldsymbol{X}(t_0)}{\partial \kappa} + \mathcal{O}(\Delta\kappa^2) \tag{28}$$

$$\boldsymbol{y}(t_1) = \boldsymbol{\Phi}(t_1, t_0)\frac{\partial \boldsymbol{X}(t_0)}{\partial \kappa}\Delta\kappa + \mathcal{O}(\Delta\kappa^2) \tag{29}$$

$$\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} \equiv \lim_{\Delta\kappa \to 0} \frac{\boldsymbol{Y}(t_1) - \boldsymbol{Y}^*(t_1)}{\Delta\kappa} \equiv \lim_{\Delta\kappa \to 0} \frac{\boldsymbol{y}(t_1)}{\Delta\kappa} \tag{30}$$

$$\frac{\partial \boldsymbol{X}(t_1)}{\partial \kappa} = \boldsymbol{\Phi}(t_1, t_0)\frac{\partial \boldsymbol{X}(t_0)}{\partial \kappa} \tag{31}$$

Discontinuites in the Spacecraft State

Equation (31) can be used to determine the derivative of the spacecraft state just before a flyby with respect to a parameter κ, which affects the spacecraft state at the initial time. Because the partial derivative of the spacecraft state and costate after the flyby with respect to the spacecraft state before the flyby is known analytically from Eqs. (11) and (17), one can use the chain rule to compute the derivative of the spacecraft state after the flyby with respect to κ. Using the chain rule and Eq. (31) allows the computation of the partial derivative of the spacecraft state at any time with respect to a parameter κ that affects the spacecraft state at any other time. For example, if κ affects the spacecraft state at t_3 and the spacecraft state is discontinuous at t_4 then the derivative of the spacecraft state at t_5 with respect to κ is determined from Eq. (32). Equation (32) is valid only if the equations of motion are continuous between t_3 and t_{4-} and between t_{4+} and t_5 and the partial derivative of the spacecraft state at t_{4+} with respect to the state at t_{4-} is known analytically. The discontinuity at t_4 need not be because of a flyby. Equation (32) is valid for any discontinuity where the partial derivative of the state after the discontinuity with respect to the state before the discontinuity is known.

$$\frac{\partial \boldsymbol{X}(t_5)}{\partial \kappa} = \boldsymbol{\Phi}(t_5, t_{4+})\frac{\partial \boldsymbol{X}(t_{4+})}{\partial \boldsymbol{X}(t_{4-})}\boldsymbol{\Phi}(t_{4-}, t_3)\frac{\partial \boldsymbol{X}(t_3)}{\partial \kappa} \tag{32}$$

Equation (32) and the chain rule can be used to determine the partial derivative of any cost or constraint function C, which is a function of only the state at a particular time and the time. The derivative provided in Eq. (33) is accurate to the accuracy of the integrator used to integrate the equations of motion of the state vector and the state transition matrix.

$$\frac{\partial C[\boldsymbol{X}(t), t]}{\partial \kappa} = \left(\frac{\partial C}{\partial \boldsymbol{X}}\right)_{X = X^*(t)} \frac{\partial \boldsymbol{X}(t)}{\partial \kappa} \tag{33}$$

Partial Derivatives with Respect to Time Parameters

If the parameter with which the partial derivatives are desired is the time of a flyby or the initial time, a slightly different method must be used because the state transition matrix can map only perturbations that occur at a common time. To use this method, one must determine the value of the state at $t_{\text{fb}+}$ if the flyby occurs at the nominal value of t_{fb} and what the value of the spacecraft state would be at $t_{\text{fb}+}$ if the flyby occurs at $t_{\text{fb}} + \Delta t_{\text{fb}}$ in the limit as Δt_{fb} goes to zero. The value of this difference is used in Eq. (34) to determine the partial derivative $\{\partial/\partial t_{\text{fb}}[\boldsymbol{X}(t_{\text{fb}})]\}_{\&}$ mapped to the common time t_{fb}. In Fig. 6, subscript 0 denotes t_{fb}, and subscript 1 denotes $t_{\text{fb}} + \Delta t_{\text{fb}}$. $\boldsymbol{X}_{A0}$ is the nominal value of the spacecraft state after a flyby occurring at t_{fb}. $\boldsymbol{X}_B$ denotes the path the spacecraft would traverse if the flyby did not occur until $t_{\text{fb}} + \Delta t_{\text{fb}}$. $\boldsymbol{X}_{C1}$ is what the spacecraft state would be immediately after a flyby if the flyby occurred at $t_{\text{fb}} + \Delta t_{\text{fb}}$. $\boldsymbol{X}_{C0}$ is the value the spacecraft state would have at t_{fb} if the spacecraft state equations of motion were integrated backward in time from $\boldsymbol{X}_{C1}$ at $t_{\text{fb}} + \Delta t_{\text{fb}}$ to t_{fb}.

$$\left[\frac{\partial \boldsymbol{X}(t_{\text{fb}})}{\partial t_{\text{fb}}}\right]_{\&} = \lim_{\Delta t_{\text{fb}} \to 0} \frac{\boldsymbol{X}_{C0} - \boldsymbol{X}_{A0}}{\Delta t_{\text{fb}}} \tag{34}$$

The spacecraft position and the cost function are continuous across a flyby, and the velocity of the spacecraft is changed only by the $\Delta\boldsymbol{v}$ from the flyby, which is a function of the time of the flyby because the velocity of the planet providing the gravity assist varies with time.

$$\boldsymbol{r}_{B0} = \boldsymbol{r}_{A0} \tag{35}$$

$$\boldsymbol{v}_{B0} = \boldsymbol{v}_{A0} - \Delta\boldsymbol{v}(t_{\text{fb}}) \tag{36}$$

$$J_{B0} = J_{A0} \tag{37}$$

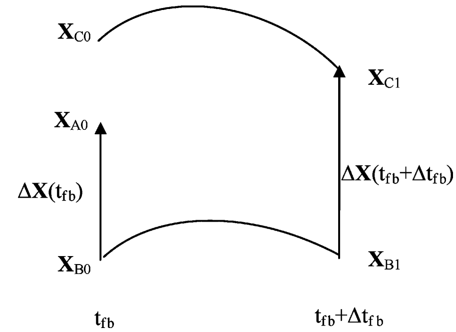

Fig. 6 Derivatives with respect to time.

Using a Taylor series about the time of the flyby, $\boldsymbol{X}_{B1}$ can be written as

$$\boldsymbol{X}_{B1} = \boldsymbol{X}_{B0} + \boldsymbol{X}'_{B0}\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{38}$$

By definition,

$$\boldsymbol{r}_{C1} = \boldsymbol{r}_{B1} \tag{39}$$

$$\boldsymbol{v}_{C1} = \boldsymbol{v}_{B1} + \Delta\boldsymbol{v}(t_{\text{fb}} + \Delta t_{\text{fb}}) \tag{40}$$

$$J_{C1} = J_{B1} \tag{41}$$

Again using a Taylor series,

$$\Delta\boldsymbol{v}(t_{\text{fb}} + \Delta t_{\text{fb}}) = \Delta\boldsymbol{v}(t_{\text{fb}}) + \frac{\mathrm{d}\Delta\boldsymbol{v}(t_{\text{fb}})}{\mathrm{d}t}\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{42}$$

It is noted that $\boldsymbol{\lambda}$ is discontinuous across the flyby with the parameters $\alpha, \gamma, \alpha', \gamma', \lambda_v(t_{\text{fb}+})$, and $\lambda'_v(t_{\text{fb}+})$ held fixed in the computation of the partial derivative. As a result, using another Taylor-series expansion, this time about the position and velocity of the spacecraft on the nominal trajectory at the time of the flyby, one obtains the value of the costates in Eq. (43). Note the higher-order terms in Eqs. (43) and (44) are $\mathcal{O}(\Delta t_{\text{fb}}^2)$ because $\boldsymbol{X}_{C1} - \boldsymbol{X}_{A0}$ is $\mathcal{O}(\Delta t_{\text{fb}})$. The partial derivatives in Eqs. (43), (44), and (49) are evaluated at $\boldsymbol{X} = \boldsymbol{X}_{A0}$.

$$\boldsymbol{\lambda}_{C1} = \boldsymbol{\lambda}_{A0} + \left(\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}(\boldsymbol{r}_{C1} - \boldsymbol{r}_{A0}) + \left(\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}(\boldsymbol{v}_{C1} - \boldsymbol{v}_{A0}) + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{43}$$

$$\boldsymbol{X}'_{C1} = \boldsymbol{X}'_{A0} + \left(\frac{\partial\boldsymbol{X}'}{\partial\boldsymbol{X}}\right)_{X = X_{A0}}(\boldsymbol{X}_{C1} - \boldsymbol{X}_{A0}) + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{44}$$

$$\boldsymbol{X}_{C0} = \boldsymbol{X}_{C1} - \boldsymbol{X}'_{C1}\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{45}$$

Equations (35–45) can be used to determine $\boldsymbol{X}_{C0}$,

$$\boldsymbol{r}_{C0} = \boldsymbol{r}_{A0} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{46}$$

$$\boldsymbol{v}_{C0} = \boldsymbol{v}_{A0} + \left[\frac{\mathrm{d}\Delta\boldsymbol{v}(t_{\text{fb}})}{\mathrm{d}t} + \boldsymbol{v}'_{B0} - \boldsymbol{v}'_{A0}\right]\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{47}$$

$$J_{C0} = J_{A0} - J'_{A0}\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{48}$$

$$\boldsymbol{\lambda}_{C0} = \boldsymbol{\lambda}_{A0} + \left[\left(\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}\boldsymbol{v}_{B0} + \left(\frac{\partial\boldsymbol{\lambda}}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}\boldsymbol{v}'_{B0} - \boldsymbol{\lambda}'_{A0}\right]\Delta t_{\text{fb}} + \mathcal{O}\left(\Delta t_{\text{fb}}^2\right) \tag{49}$$

Combining Eqs. (34) and (46) to (49) yields the partial derivative mapped to the time of the flyby. These partial derivatives in Eq. (50) can be used to compute the partial derivative of the state at any time with respect to the time of the flyby by using Eq. (32). Note Eq. (50) should only be used if the state transition matrix will be used to map the derivative from the time of the flyby to a future common time. For example, Eq. (51) could be used to calculate the derivative of the spacecraft state at the final time with respect to the time of the flyby. If the partial derivative of the spacecraft state at the time of the flyby with respect to the time of the flyby is desired, one can use the normal method of calculating a partial derivative and obtain Eq. (52). The method to obtain the partial derivative of the state with respect to the initial time can be computed in the same way. The only change in the value of the partial derivative is that the term $\mathrm{d}/\mathrm{d}t\,(\Delta\boldsymbol{v})$ is zero because there is no impulsive $\Delta\boldsymbol{v}$ at t_0.

$$\left[\frac{\partial\boldsymbol{X}(t_{\text{fb}+})}{\partial t_{\text{fb}}}\right]_{\&} = \begin{bmatrix} \boldsymbol{0} \\ \dfrac{\mathrm{d}\Delta\boldsymbol{v}(t_{\text{fb}})}{\mathrm{d}t} + \boldsymbol{v}'_{B0} - \boldsymbol{v}'_{A0} \\ J'_{B0} - J'_{A0} \\ \left(\dfrac{\partial\boldsymbol{\lambda}_r}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}\boldsymbol{v}_{B0} + \left(\dfrac{\partial\boldsymbol{\lambda}_r}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}\boldsymbol{v}'_{B0} - \boldsymbol{\lambda}'_{r_{A0}} \\ \left(\dfrac{\partial\boldsymbol{\lambda}_v}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}\boldsymbol{v}_{B0} + \left(\dfrac{\partial\boldsymbol{\lambda}_v}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}\boldsymbol{v}'_{B0} - \boldsymbol{\lambda}'_{v_{A0}} \\ 0 \end{bmatrix} \tag{50}$$

$$\frac{\partial\boldsymbol{X}(t_f)}{\partial t_{\text{fb}}} = \boldsymbol{\Phi}(t_f, t_{\text{fb}+})\left[\frac{\partial\boldsymbol{X}(t_{\text{fb}+})}{\partial t_{\text{fb}}}\right]_{\&} \tag{51}$$

$$\frac{\partial\boldsymbol{X}(t_{\text{fb}})}{\partial t_{\text{fb}}} = \lim_{\Delta t_{\text{fb}} \to 0}\frac{\boldsymbol{X}_{C1} - \boldsymbol{X}_{A0}}{\Delta t_{\text{fb}}} = \begin{bmatrix} \boldsymbol{v}_{B0} \\ \dfrac{\mathrm{d}\Delta\boldsymbol{v}(t_{\text{fb}})}{\mathrm{d}t} + \boldsymbol{v}'_{B0} \\ J'_{B0} \\ \left(\dfrac{\partial\boldsymbol{\lambda}_r}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}\boldsymbol{v}_{B0} + \left(\dfrac{\partial\boldsymbol{\lambda}_r}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}\boldsymbol{v}'_{B0} \\ \left(\dfrac{\partial\boldsymbol{\lambda}_v}{\partial\boldsymbol{r}}\right)_{X = X_{A0}}\boldsymbol{v}_{B0} + \left(\dfrac{\partial\boldsymbol{\lambda}_v}{\partial\boldsymbol{v}}\right)_{X = X_{A0}}\boldsymbol{v}'_{B0} \\ 0 \end{bmatrix} \tag{52}$$

Impulsive Δv Model Example

It is desired to compute the minimum fuel trajectory from Earth to Saturn using three Venus gravity assists and the impulsive $\Delta\boldsymbol{v}$ model. The trajectory is shown in Fig. 7. There are 12 equality constraints requiring the spacecraft to intercept Venus at the time of each flyby and Saturn at the final time and three inequality constraints requiring the flyby periapse to be greater than or equal to 6400 km for each of the gravity-assist maneuvers. There are 29 free parameters. The partial derivatives (a total of 377) of the cost and equality constraints with respect to each of the 29 parameters are needed in order to determine the optimal values of these parameters. The derivatives of the inequality constraints with respect to the free parameters are trivial and can be calculated analytically.

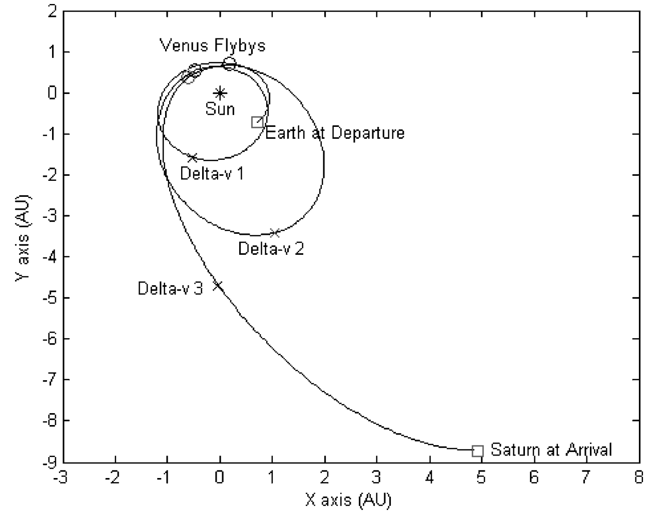

Fig. 7 Impulsive Δv Earth-Venus-Venus-Venus-Saturn trajectory.

Table 1 provides a sample of the partial derivatives of the constraints with respect to the free parameters. The analytic solution is obtained by propagating the orbit using Kepler's equation. The state transition matrix for the orbital elements of a spacecraft in a central body force field is available analytically from Eq. (53). The analytic state transition matrix for the spacecraft state can be obtained from Eq. (54). The state transition matrix derivatives agree with the analytic derivatives to at least six digits while the derivatives obtained using the optimal perturbation step agree only to at least four digits. The central difference and forward difference derivatives calculated using a perturbation step size of 10^{-4} and 10^{-6} respectively do not all agree with the analytic solution to more than one significant figure. As a result, one must tune the step sizes using either Hull's method or another technique in order to use finite difference derivatives. The state transition matrix derivatives do not require any tuning.

Table 1 Accuracy of partial derivatives for the impulsive Δv trajectory

Derivative type	$\partial y_f/\partial \alpha_{\mathrm{pfb1}}$, AU[a]	$\partial x_{\mathrm{fb3}}/\partial r_{m1}$, AU/km	$\partial x_f/\partial r_{m1}$, AU/km	$\partial x_{\mathrm{fb1}}/\partial \Delta v_0$, days	$\partial y_f/\partial \Delta v_0$, days
Analytic solution	14.01651	-1.248655×10^{-3}	$1.11043871 \times 10^{-2}$	−187.41078837	11746.297
STM[b]	14.01649	-1.248648×10^{-3}	$1.11043892 \times 10^{-2}$	−187.41078844	11746.305
Accurate Digits STM	6	6	7	10	6
CD*[c]	14.01839	-1.243751×10^{-3}	1.103744×10^{-2}	−187.4135	4738.943
CD* Accurate digits	4	3	2	5	0
FD**[d]	14.01537	-1.137700×10^{-3}	8.662536×10^{-3}	−187.4205	5186.494
FD** Accurate digits	4	1	1	4	1
CD***[e]	14.01846	-1.248639×10^{-3}	1.110478×10^{-2}	−187.4107	11746.69
CD*** Accurate digits	4	5	5	6	5
CD*** Perturbations	1.873×10^{-4}	7.810×10^{-1}	2.995×10^{-3}	3.533×10^{-7}	8.830×10^{-10}

[a] AU = astronomical unit.
[b] STM = state transition matrix.
[c] CD* = central difference derivatives with a perturbation size of 10^{-4}.
[d] FD** = forward difference derivatives with a perturbation size of 10^{-6}.
[e] CD*** = central difference derivatives with the perturbation size given by Hull's method.[11]

Table 2 Optimal values of free parameters for the impulsive Δv trajectory

Parameters	Initial estimate	Converged central difference values	Converged state transition matrix values
t_0, Julian date	2,455,414.500	2,455,415.572	2,455,415.983
t_f, Julian date	2,459,066.500	2,459,095.251	2,459,084.386
$v_{\infty 0}$, km/s	3.0932	3.0643	3.0657
α_0, radians	0.02866	−0.00571	−0.01677
γ_0, radians	3.170370	3.15983	3.15856
t_{fb1}, Julian date	2,455,522.500	2,455,523.153	2,455,523.226
r_{m1}, km	6400	6437.742	6400.000
β_1, radians	6.28	6.2433	6.2453
Δv_{pow1}, m/s	10	−0.002133	2.1315×10^{-7}
t_{pfb1}, Julian date	2,455,735.500	2,455,740.630	2,455,742.318
Δv_{pfb1}, km/s	1.048949	0.907019	0.913924
α_{pfb1}, radians	2.80579	3.18473	3.15633
γ_{pfb1}, radians	−0.08727	0.00228	0.00151
t_{fb2}, Julian date	2,456,006.500	2,456,008.088	2,456,008.100
r_{m2}, km	6400	6400.000	6400.000
β_2, radians	6.28	6.2990	6.3385
Δv_{pow2}, m/s	10	9.149208	19.573638
t_{pfb2}, Julian date	2,456,532.500	2,456,586.529	2,456,571.467
Δv_{pfb2}, km/s	0.440677	0.251181	0.254642
α_{pfb2}, radians	2.570033	3.15411	3.20358
γ_{pfb2}, radians	−0.00713	0.03142	0.10534
t_{fb3}, Julian date	2,457,141.500	2,457,141.311	2,457,141.343
r_{m3}, km	6400	6400.000	6400.000
β_3, radians	6.28	6.7057	6.7079
Δv_{pow3}, m/s	10	0.00458442	2.901430×10^{-7}
t_{pfb3}, Julian date	2,457,532.500	2,457,540.164	2,457,536.620
Δv_{pfb3}, km/s	2.562117	0.188114	0.167205
α_{pfb3}, radians	2.11001	−0.38073	−0.42447
γ_{pfb3}, radians	0.20579	1.44400	1.43881
Cost, km/s	7.174943	4.419799	4.421049
Computational time, s	N/A	489	28
$\lvert \boldsymbol{r}(t_{\mathrm{fb1}}) - \boldsymbol{r}_{p1}(t_{\mathrm{fb1}})\rvert$, m	N/A	1927.118	0.25507
$\lvert \boldsymbol{r}(t_{\mathrm{fb2}}) - \boldsymbol{r}_{p2}(t_{\mathrm{fb2}})\rvert$, m	N/A	134,230.175	13.66192
$\lvert \boldsymbol{r}(t_{\mathrm{fb3}}) - \boldsymbol{r}_{p3}(t_{\mathrm{fb3}})\rvert$, m	N/A	1,239,144.187	37.01409
$\lvert \boldsymbol{r}(t_f) - \boldsymbol{r}_{\mathrm{pf}}(t_f)\rvert$, m	N/A	19,937,981.480	777.743842

Table 3 Accuracy of partial derivatives for Earth-Venus-Venus-Saturn PLP trajectory

Derivative type	$\partial J/\partial \gamma_0'$, AU^2/day^2	$\partial x_{\text{fb1}}/\partial \lambda_{v0}'$, day^3	$\partial z_f/\partial \lambda_{v0}'$, day^3	$\partial y_f/\partial \alpha_0$, AU	$\partial x_f/\partial \alpha_1'$, AU × days
STM	1.94263×10^{-4}	4.70782×10^7	7.36889×10^{10}	38075.09	2243.230
CD*	2.42869×10^{-4}	4.70795×10^7	7.36838×10^{10}	36750.53	2271.388
CD* accurate digits	1	4	4	1	2
FD**	1.97418×10^{-4}	4.72702×10^7	7.02872×10^{10}	38057.03	2229.010
FD** accurate digits	2	2	1	3	2
CD***	1.94270×10^{-4}	4.70782×10^7	7.36923×10^{10}	38062.61	2243.490
CD*** Accurate digits	4	6	4	3	4
CD*** Perturbations	1.286×10^{-7}	7.437×10^{-12}	1.835×10^{-13}	9.946×10^{-8}	1.537×10^{-7}

[a]AU = astronomical unit.
[b]STM = state transition matrix.
[c]CD* = central difference derivatives with a perturbation size of 10^{-4}.
[d]FD** = forward difference derivatives with a perturbation size of 10^{-6}.
[e]CD*** = central difference derivatives with the perturbation size given by Hull's method.[11]

Table 4 Optimal values of free parameters for Earth-Venus-Saturn PLP trajectory

Parameters	Initial estimate	Converged central difference values	Converged state transition matrix values
t_0, Julian date	2,456,967.358	2,456,964.820	2,456,965.025
t_f, Julian date	2,458,967.360	2,458,964.622	2,458,964.714
α_0, radians	1.60231	1.60575	1.60181
γ_0, radians	0.51606	0.42259	0.53478
α_0', rad/s	−0.32021	−0.34486	−0.26348
γ_0', rad/s	8.82879	8.83680	8.40122
λ_{v0}, AU/day^2	-9.04494×10^{-12}	8.33069×10^{-10}	1.87900×10^{-10}
λ_{v0}', AU/day^3	8.04304×10^{-8}	7.99615×10^{-8}	8.03277×10^{-8}
t_{fb1}, Julian date	2,457,828.632	2,457,826.184	2,457,826.269
r_{m1}, km	252.500	252.038	250.824
β_1, radians	6.08664	6.03326	6.02885
α_1, radians	2.78957	2.51727	1.82834
γ_1, radians	−0.306563	−0.96504	−0.16443
α_1', rad/s	−8.1872	−7.94359	−7.72772
γ_1', rad/s	−3.33312	−3.56769	−4.02256
λ_{v1}, AU/day^2	1.00000×10^{-9}	-1.76753×10^{-11}	-8.29047×10^{-12}
λ_{v1}', AU/day^3	0.00000	-1.78137×10^{-10}	-1.78889×10^{-10}
Cost, km^2/s^3	N/A	1.3082×10^{-2}	1.3081×10^{-2}
Computational time, s	N/A	19	15
$\lvert \boldsymbol{r}(t_{\text{fb1}}) - \boldsymbol{r}_{p1}(t_{\text{fb1}})\rvert$, km	3.92124×10^6	3.54656	0.14097
$\lvert \boldsymbol{r}(t_f) - \boldsymbol{r}_{\text{pf}}(t_f)\rvert$, km	3.70381×10^8	9772.62944	88.82233

$$\boldsymbol{\Phi}_\alpha(t, t_0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\dfrac{3\sqrt{\mu}(t - t_0)}{2a^{2.5}} & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{53}$$

$$\boldsymbol{\Phi}(t, t_0) = \frac{\partial \boldsymbol{X}(t)}{\partial \boldsymbol{Z}(t)} \boldsymbol{\Phi}_\alpha(t, t_0) \frac{\partial \boldsymbol{Z}(t_0)}{\partial \boldsymbol{X}(t_0)} \tag{54}$$

The central difference derivatives require one to determine the optimal perturbation step size for each constraint with respect to each parameter. Note in Table 1 that the optimal perturbation step size for Δv_0 to compute the derivative of the spacecraft's x position at the time of the first flyby with respect to Δv_0 is three orders of magnitude smaller than the optimal Δv_0 perturbation to calculate the derivative of the spacecraft's final y position with respect to Δv_0. Because the computational time necessary to perturb the free parameters and integrate the trajectory 754 times is large, a single perturbation size for each parameter must be determined, which will allow the derivatives of all constraints with respect to that parameter to be calculated as accurately as possible.

The state transition matrix derivatives require significantly less computational time than the central difference derivatives for this example. The state transition matrix derivatives are faster because they require the trajectory to be integrated only one time using 42 equations. The central difference derivatives require six equations to be integrated 59 times each for a total of 354 equation integrations. The time required to use the central difference derivatives can be reduced slightly through the use of nodes in the calculation of the central difference derivatives.

Table 2 provides the optimized values for each of the free parameters obtained using a sequential-quadratic-programming algorithm.* The same initial estimate of the free parameters was used to optimize the parameters using both a central difference derivative method with derivatives tuned using Hull's method[11] and a state transition matrix method. The costs using both methods are within three-hundredths of 1%. To determine which method could satisfy the constraints to a greater accuracy, the convergence criteria were made increasingly stringent until the method failed to converge. The state transition method was able to satisfy the constraints to four orders of magnitude more than the central difference method. The central difference method also took over 17 times more computational time than the state transition matrix method. All computational times were determined on a 2.4-GHz Pentium 4® processor.

*Data available online at http://hsl.rl.ac.uk/archive2002/hslarchive/packages/vfl3/vfl3.pdf [cited 29 July 2003].

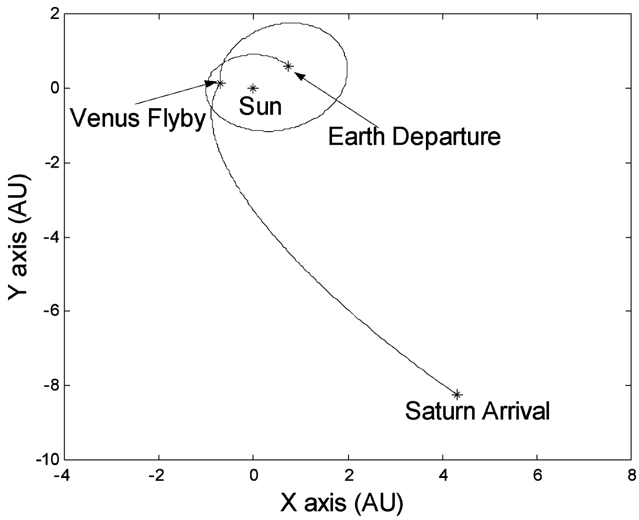

Fig. 8 Earth-Venus-Saturn PLP trajectory.

Finite-Thrust Model Example

An optimal trajectory using PLP from Earth to Saturn with two gravity assists provided by Venus is used to demonstrate the accuracy of the state transition matrix derivatives for the finite thrust model. Table 3 provides a sample of the derivatives of the costs and constraints with respect to the free parameters. Again the state transition matrix derivatives agree closely with the optimal central difference derivatives without requiring any work to determine the optimal perturbation step sizes. In this case, there is no theoretical value for the derivatives, but the state transition matrix derivatives are accurate to the accuracy of the integrator's ability to integrate the equations of motion for the spacecraft state and the state transition matrix. Using the rule of thumb given by Gill, the central difference derivatives are accurate to one-half the accurate digits determined by integrating the equations of motion of the spacecraft state.[10] As a result, the state transition matrix derivatives are considered to be the true solution when calculating the accurate digits in the finite difference derivatives. For this trajectory using the central difference derivatives to compute the derivatives requires about 40% more computational time per iteration than the state transition matrix derivatives.

To demonstrate the benefits of using the more accurate derivatives obtained from the state transition method, an optimal trajectory using PLP from Earth to Saturn with one Venus flyby is sought. The spacecraft must satisfy six equality constraints requiring it to intercept Venus at the time of the gravity assist and to intercept Saturn at the final time. The total mission time is constrained to be less than or equal to 2000 days, and the flyby radius is constrained to be greater than or equal to 250 km. This flyby radius is obviously a subsurface flyby, which is not possible in an actual mission. The small flyby radius is used to create a single gravity-assist trajectory that has a lower cost than a simple trajectory from Earth to Saturn with no flyby. Obviously, a flyby of Venus is not beneficial for a 2000-day Earth-to-Saturn intercept mission using a PLP engine. The trajectory is shown in Fig. 8. Table 4 shows a comparison of the optimal trajectories computed using both central difference and state transition matrix derivatives. Like the impulsive Δv case, the cost is nearly the same for both solutions, but the state transition matrix derivatives are able to satisfy the constraints 100 times better and require significantly less computational time than the central difference derivatives.

Conclusions

The state transition matrix method provides a tool to calculate the derivatives of the cost and constraint functions with respect to the free parameters for trajectories where the equations of motion of the spacecraft are continuous. The spacecraft state might be discontinuous a finite number of places so long as the partial derivative of the spacecraft state after the discontinuity with respect to the spacecraft state before the discontinuity is known analytically. The state transition matrix derivatives are shown to be orders of magnitude more accurate than the finite difference derivatives. They are also able to satisfy the constraints for the two example problems to orders of magnitude greater accuracy. The state transition matrix derivatives offer a significant reduction in computational time compared to the central difference derivatives. The main disadvantage of using the state transition matrix to calculate derivatives is that it requires numerous analytic partial derivatives to be determined and coded. Depending on the problem being solved and the number of solutions sought, the computational time saved by using state transition matrix derivatives might or might not offset the additional time required to derive the analytic partial derivatives.

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